CLASSICAL CASES OF THE INDEX THEOREM: EULER OPERATOR

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Abstract. This is an expository article on geometry. As a special case of the Atiyah-Singer index theorem, I calculate the index of the Euler operator.

Contents

1. Introduction

The index of the Euler operator is an interesting special case of the index theorem which can be proved directly by Hodge theory. It is equal to the Euler characteristic χ, which is a topological invariant.

2. Atiyah-Singer geometric cases: Euler operator

I discuss the index theorem for certain standard geometric operators, starting with the Euler operator. In a future article, I will discuss the relationship with the Gauss-Bonnet theorem.

Let M a smooth manifold. Define the *Euler operator* as the operator $d + d^*$, on differential forms, but restricted to mapping even forms to odd forms.

We write $\mathcal{D} = (d+d^*)^{\text{ev}}$, where the ev superscript indicates the restriction to even forms. This notation indicates the generalisation to Dirac operators.

$$\mathcal{D} : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M) \tag{1}$$

The formal adjoint is $\mathcal{D}^* = ((d + d^*)^{\text{ev}})^* = (d + d^*)^{\text{odd}}$, the restriction to odd forms mapping in the other direction.

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Define the *index* of $\mathcal{D}$ as follows, briefly. The genesis of the problem will be covered in a future article. $\mathcal{D}$ has finite-dimensional kernel and cokernel. So we can define the index as the difference of their dimensions.

$$\operatorname{index}\mathcal{D} = \dim\operatorname{Ker}\mathcal{D} - \dim\operatorname{Coker}\mathcal{D} \tag{2}$$

Because $\operatorname{Coker}\mathcal{D} = \operatorname{Ker}\mathcal{D}^*$, rewrite the index as

$$\operatorname{index}\mathcal{D} = \dim\operatorname{Ker}\mathcal{D} - \dim\operatorname{Ker}\mathcal{D}^* \tag{3}$$

Recall that the square of $d + d^*$ is the *Hodge Laplacian* on forms, $\Delta = dd^* + d^*d$. Define *harmonic forms* as $\mathcal{H}^* = \operatorname{Ker}\Delta$.

Proposition 1 (Reduction to Dirac case)**.**

$$\operatorname{Ker}\Delta = \operatorname{Ker}\mathcal{D} \tag{4}$$

Proof. See [1] (13.18 prec.)

This result changes the problem from second degree to first degree. Furthermore, this is compatible with the grading. Restricting operators now to forms of degree k, we have

Theorem 1 (Main theorem of Hodge theory)**.** *The space of harmonic k-forms is isomorphic to the de Rham cohomology space of degree k.*

$$\mathcal{H}^k(M) \ \simeq H^k(M). \tag{5}$$

Proof. See [1] (13.18)

Theorem 2 (Euler operator)**.** *The Euler characteristic is equal to the index of the Euler operator.*

Proof by Hodge theory. (See [1], (13.20))

Write the Euler characteristic χ as the difference of the dimension of the even and odd degree cohomology, which is a step to reframe the situation as:

$$\chi(M) = \sum_{k \text{ even}} \dim H^k(M) - \sum_{k \text{ odd}} \dim H^k(M) \tag{6}$$

By Hodge theory above (5), the cohomology vector space H^* is isomorphic to the space of harmonic forms $\mathcal{H}^*$, in a manner compatible with the grading so (6) is also the difference of the dimensions of even and odd harmonic forms.

$$\chi(M) = \sum_{k \text{ even}} \dim \mathcal{H}^k(M) - \sum_{k \text{ odd}} \dim \mathcal{H}^k(M) \tag{7}$$

Now use the fact that the space of harmonic forms is the kernel of the Euler operator, (4).

Recalling that the Euler operator is the restriction to even forms, the first term is dim $\operatorname{Ker}\mathcal{D}$, and the second term dim $\operatorname{Ker}\mathcal{D}^*$, and the difference is the index (3).

Therefore we have proved that the Euler characteristic of the manifold, is the index of the Euler operator.

3. Conclusion

Geometric problems on the solutions of equations can be solved by the index theorem.

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References

[1] Bleecker, Booss-Bavnbek *Index theory with Applications to Mathematics and Physics* International Press (2013)
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[2] D. Eisenbud, J. Harris *3264 and all that. A second course in algebraic geometry* Cambridge University Press (2016)

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